UBC Math Circle 2018 Problem Set 1

I. Introductory Problems

1. Prove that $2 + 3 + 5 + \cdots + (2n+1) = n^2$ for all natural numbers n.

> **Solution:**
>
> For $n = 1$, $1 = 1^2$.
>
> Assume the statement is true for some natural number k.
>
> Let $n = k + 1$. We have
>
> $$\begin{aligned} 1 + 3 + 5 + \cdots + [2(k+1) - 1] &= [1 + 3 + 5 + \cdots + (2k-1)] = (2k+1) \\ &= k^2 + (2k+1) \\ &= (k+1)^2 \end{aligned}$$

2. Prove that $23^n - 1$ is divisible by 11 for all positive integers n.

> **Solution:**
>
> Let $n = 1$. $23^1 - 1 = 22$, which is divisible by 11.
>
> Assume $11 | 23^k - 1$ for some positive integer k.
>
> Then, $23^{k+1} - 1 = 23 \cdot 23^k - 1 = 11 \cdot 2 \cdot 23^k + (23^k - 1)$, which is also divisible by 11.

II. Intermediate Problems

3. Prove that for every positive integer n, one of the numbers n, $n+1$, $n+2$, $\ldots$, $2n$ is the square of an integer.

> **Solution:** Let $n = 1$. Then the sequence $(n, n+1, n+2, \ldots, 2n)$ is $(1, 2)$. 1 is the square of itself, so the sequence contains the square of an integer.
>
> Assume one of the numbers in the sequence $(m, m+1, m+2, \ldots, 2m)$ is the square of an integer for $m > n$. Consider the sequence $(m+1, m+2, \ldots, 2m, 2m+1, 2(m+1))$. Since the first sequence contains a square number, there are two possible cases to consider:

1. A number in the subsequence $(m+1, m+2, \ldots, 2m)$ is square

   This subsequence is contained in the second sequence, $(m+1, m+2, \ldots, 2m, 2m+1, 2(m+1))$ and hence the second sequence also contains a square number.

2. No number in the subsequence $(m+1, m+2, \ldots, 2m)$ is square

   In this case, m must be a square number. Note that $\sqrt{m} \geq 2$ since $m > n = 1$ and we have assumed that $\sqrt{m}$ is an integer. Then, the distance between m and the next square number $(\sqrt{m}+1)^2$ satisfies

   $$(\sqrt{m}+1)^2 - (\sqrt{m})^2 = m + 2\sqrt{m} + 1 - m = 2\sqrt{m} + 1 \leq (\sqrt{m})^2 + 1 = m + 1$$

   Hence, $(\sqrt{m}+1)^2 \leq 2m+1$, so $(\sqrt{m}+1)^2$ is contained in the new sequence $(m+1, m+2, \ldots, 2m, 2m+1, 2(m+1))$.

It was shown that the sequence $(m+1, m+2, \ldots, 2m, 2m+1, 2(m+1))$ contains a square number in both cases above for $m > n = 1$. Hence, one of the numbers $(k, k+1, \ldots, 2k)$ is the square of an integer for all $k \geq 1$.

4. Show for $n \geq 1$

$$\sum_{i=0}^{n} i(3)^i = \frac{3^{n+1}(2n-1)+3}{4}.$$

Solution: Assume the statement holds for $1 \leq n \leq k$. Let $n = k + 1$:

$$
\begin{aligned}
\sum_{i=0}^{n+1} i3^i &= 0(3)^0 + 1(3)^1 + \cdots + n(3)^n + (n+1)(3)^{n+1} \\
&= \sum_{i=0}^{n} i3^i + (k+1)(3)^{k+1} \\
&= \frac{3^{k+1}(2k-1)+3}{4} + (k+1)(3)^{k+1} \\
&= \frac{3^{k+1}(2k-1)+3+4(k+1)(3)^{k+1}}{4} \\
&= \frac{3^{k+1}(2k-1+4k+4)+3}{4} \\
&= \frac{3^{k+1}(6k+3)+3}{4} \\
&= \frac{3^{k+1}(3)(2k+1)+3}{4} \\
&= \frac{3^{k+2}(2(k+1)-1)+3}{4}
\end{aligned}
$$

5. Consider the following exponential tower.

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$$

Let $a_0 = 1$ and $a_{n+1} = \sqrt{2}^{a_n}$. Prove that the sequence a_n is monotonically increasing and bounded above.

Solution: By solving the equation $a = \sqrt{2}^a$ we can see that 2 is the limiting value of the sequence.

We can induct on n, with the hypothesis that $a_n < 2$.

Fun fact, sequences of the form $b_{n+1} = b^{b_n}$ only converge for numbers b in the following range.

$$0.065988... = e^{-e} \leq b \leq e^{1/e} = 1.44466...$$

III. Advanced Problems

6. Let F_{i+1} denote the $(i+1)^{th}$ Fibonacci number (ie. $F_{i+1} = F_i + F_{i-1}$). Prove using induction:

$$F_{2n} = F_n(F_{n+1} + F_{n-1})$$
$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

Solution: Base case, $n = 0$:

$$F_{2n} = F_2 = 1 = 1(1+0) = F_1(F_2 + F_0)$$

$$F_{2n+1} = F_3 = F_2 = 1^2 + 1^2 = F_2^2 + F_1^2$$

Assume $F_{2n} = F_n(F_{n+1} + F_{n-1})$ and $F_{2n+1} = F_{n+1}^2 + F_n^2$ hold for $1 \leq n \leq k$. Let $n = k+1$:

$$\begin{aligned}
F_{2(k+1)} &= F_{2k+2} \\
&= F_{2k+1} + F_{2k} \\
&= F_{k+1}^2 + F_k^2 + F_k(F_{k+1} + F_{k-1}) \\
&= F_{k+1}^2 + F_k^2 + F_kF_{k+1} + F_kF_{k-1} \\
&= F_k(F_k + F_{k-1}) + F_{k+1}(F_{k+1} + F_k) \\
&= F_kF_{k+1} + F_{k+1}F_{k+2} \\
&= F_{k+1}(F_{k+2} + F_k)
\end{aligned}$$

$$\begin{aligned}
F_{2(k+1)+1} &= F_{2k+3} \\
&= F_{2k+2} + F_{2k+1} \\
&= F_{k+1}(F_{k+2} + F_k) + F_{k+1}^2 + F_k^2 \\
&= F_{k+1}(F_{k+1} + 2F_k) + F_{k+1}^2 + F_k^2 \\
&= F_{k+1}^2 + 2F_kF_{k+1} + F_k^2 + F_{k+1}^2 \\
&= (F_{k+1} + F_k)^2 + F_{k+1}^2 \\
&= F_{k+2}^2 + F_{k+1}^2
\end{aligned}$$

7. A knight is sitting on a black square lies on an infinite chessboard. How many squares can the knight reach after exactly n moves?

Solution: Let $f(n)$ be the number of squares the knight can be on after exactly n moves. Observe that the set of squares must be all white or all black.

By hand, we can compute $f(0) = 1$, $f(1) = 8$, and $f(2) = 33$. For $n = 3$, the knight can reach all white squares of an octagon with 4 white squares on each side. By induction, you can show that for a general n, the squares the knight can reach form an octagon of $n+1$ black or white squares on each side (depending on whether n is odd or even).

To count the number of squares in the octagon, let's first consider when n is even. All black reachable tiles lie in a square with $((4n+1)^2+1)/2$ black squares inside it. Then we can subtract off each of the corners of the squares that we've overcounted. By induction or by arithmetic series, the number of black squares has the following form.

$$4[(n-1)+(n-3)+\cdots 1] = n^2$$

Hence, $f(n)$ for even n is as follows.

$$\frac{(4n+1)^2+1}{1} - n^2 = 7n^2+4n+1$$

By a similar argument, this formula also holds for the white cells.

Thus, we obtain our expression for $f(n)$.

$$f(n) = \begin{cases} 1 & \text{when } n = 0 \\ 8 & \text{when } n = 1 \\ 33 & \text{when } n = 2 \\ 7n^2+4n+1 & \text{when } n \geq 3 \end{cases}$$

8. Foster and Tom are playing a game with a pile of $N \geq 1$ stones. They take turns removing a number of stones from the pile until the pile becomes empty. Whoever took the last stone wins the game. Foster will make the first move, and take a_1 stones, where $1 \leq a_1 \leq N-1$. Tom then takes a_2 stones from the remaining pile, where $1 \leq a_2 \leq 2a_1 - 1$. On the ith move, the player can take a_i stones from the pile, where $1 \leq a_i \leq 2a_{i-1} - 1$. Determine the values of N where Tom is guaranteed to win.

Solution: Hint: Look at powers of 2.

We prove by induction that Tom wins whenever $N = 2^k$ for some $k \geq 1$, and loses otherwise. This is equivalent to proving that Foster wins whenever $N \neq 2^k$ for some $k \geq 1$.

(Will be finished soon! (hopefully))