UBC Math Circle 2018 Problem Set 6

Big Hint: The theme this week is invariants, monotonicity, and colouring.

1. In a group of n people, each person has at most 3 enemies (if A is an enemy of B, then B is also an enemy of A). Prove that we can split the n people into two groups such that each person has at most one enemy in her group.

   Hint: Is there a way to quantify the goodness of a grouping? How can we improve the quality of a bad grouping by moving people around?

   Solution: Let S be the total of enemies in the same group. Observe that whenever we move a person from a group with two or more enemies to the other group (with at most one enemy), S strictly decreases. Also observe that S is bounded below, so we can only make a finite number of these swaps before each person only has one enemy in her group.

2. Let $x_1, x_2, ..., x_{2018}$ be integers, and $f_0(i) = x_i$. For $n > 0$, define $f_n(i)$ to be the number of integers j where $f_{n-1}(j) = f_{n-1}(i)$ (including $j = i$). Prove that there exists some integer m such that $f_m(i) = f_{m-1}(i)$ for all integers i between 1 and 2018 inclusive.

   Hint: What can we say about the number of distinct values for each f_n.

   Solution: Solution Sketch:

   Let $g(n)$ be the number of different values of $f_n(i)$ over all i. Observe that $g(n)$ is bounded below by 1, and is non-increasing. Therefore, there exists some integer k, and some integer N, such that $g(n) = k$ for all $n \geq N$.

   Suppose that $g(n-1) = g(n) = g(n+1)$, then we can show that $f_n(i) = f_{n+1}(i)$.

3. Two thousand 1×1 cells of a 2018×2018 square are infected. Each second, the cells with at least two infected neighbours become infected. Two cells are neighbours if they have a common side. Can the infection spread to the whole square?

   Solution: We observe that the perimeter of the total infected area is non-increasing. We can see this by looking at what happens when a cell becomes infected. Initially the perimeter is 4×2000, but the square has perimeter 4×2018, so the infection will never spread to the whole square.

4. We start with the number 7^{2018} (in base 10). We repeatedly remove the first digit and add it to the remaining number until we obtain a number with 10 digits. Prove that this number has two equal digits.

   Hint: What is the invariant?

Solution: It's a good guess that the digital sum is an invariant.

Observe that $7^{2018} \not\equiv 0 \pmod 9$. We can write

$$7^{2018} = d_0 + 10^1 d_1 + 10^2 d_2 + \cdots + 10^m d_m \equiv d_0 + d_1 + \cdots + d_m \pmod 9,$$

where $d_0, d_1, ...$ are the digits of 7^{2018}. Observe that this means the digital sum is invariant, because

$$d_0 + 10^1 d_1 + \cdots + 10^m d_m \equiv d_0 + 10^1 d_1 + \cdots + 10^{m-1} d_{m-1} + d_m \pmod 9.$$

Now, the sum of the ten digits in the end must also be congruent to 7^{2018} modulo 9. However, if they are all distinct, the sum will be $0 + 1 + 2 + \cdots + 9 = 45 \equiv 0 \pmod 9$, which is not equivalent to 7^{2018}.

5. A 23×23 square grid is completely tiled with 1×1, 2×2, and 3×3 tiles. What is the minimum number of 1×1 tiles needed?

   Hint: How can we colour the grid so that we can say stuff about the 2×2 and 3×3 tiles?

Solution: Observe that we could tile the square grid with at most one 1×1 tile. Put the 1×1 tile in the center, then tile each of the four 11×12 rectangles with a row of six 2×2 tiles, and three rows of 3×3 tiles.

Now we show that at least one 1×1 tile is needed. Suppose that no 1×1 tiles are needed. Colour the rows of the grid red and blue alternating, starting with red. Then we have 23 more red squares than blue ones. Each 2×2 tile covers the same number of red and blue squares. Each 3×3 tile covers three more of one colour. This means the difference between the red and blue squares should be a multiple of 3. 23 is not a multiple of three, so we can't tile the grid with only 2×2 and 3×3 tiles.

6. Rachel has n one's are written on the blackboard. Each step, Rachel erases two numbers a and b and replaces them with $\frac{a+b}{4}$. She repeats this until there is only one number left on the blackboard. Prove that this number is at least $\frac{1}{n}$.

Solution: The *AM-GM-HM* inequality gives

$$\frac{a+b}{4} \geq \frac{1}{2} \cdot \frac{2}{\frac{1}{a} + \frac{1}{b}} \implies \frac{1}{\left(\frac{a+b}{4}\right)} \leq \frac{1}{a} + \frac{1}{b}.$$

This means that the sum of the reciprocals of the numbers on the blackboard is non-increasing.

Let the final number be x. Initially the sum of reciprocals is n, so $\frac{1}{x} \leq n$. Therefore $x \geq \frac{1}{n}$.

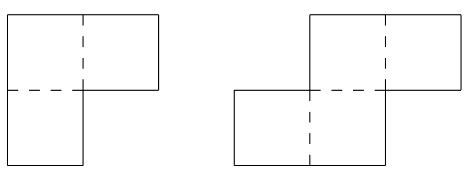
7. Eric placed m cookies at the vertices of a regular n-gon, where $m > n$. Each step, Eric chooses a vertex with at least two cookies, moves one cookie to the clockwise neighbour, and another to the counterclockwise neighbour. After k moves, the original arrangement of cookies is restored. Prove that k is a multiple of n.

Solution: Label the vertices in order from 1 to n. Let a_i be the number of times Eric chose vertex i. Observe that each time Eric chose vertex i, two cookies were removed from vertex i, and each time Eric chose one of the two vertices adjacent to vertex i, one cookie is added to vertex i. Since the original arrangement is restored, the number of cookies removed and added to each vertex must be equal. Therefore, we get the following equations

$$\begin{aligned} 2a_1 &= a_n + a_2 \\ 2a_2 &= a_1 + a_3 \\ &\vdots \\ 2a_n &= a_{n-1} + a_1 \end{aligned}$$

Without loss of generality, let a_1 be the maximum of all a_i. Then $a_2 = a_n = a_1$ from the first equation. It follows that $a_1 = a_2 = \cdots = a_n$, so the number of moves is equal to na_1, which is always a multiple of n.

8. (Putnam 2016 A4). Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown:



What is the minimum number of tiles required to tile the region?

Solution: Putnam 2016 A4.

Hint: We colour the grid red and blue such that each tile covers at most one red square. This is a lower bound on the number of tiles. We can then show that this lower bound is achievable by construction.

http://kskedlaya.org/putnam-archive/