UBC Math Circle 2018 Problem Set 7

I. Introductory Problems

1. For each of the following graphs find the degree of each vertex and determine which are isomorphic to another.

ss1.png

Solution: The top left and bottom right are isomorphic.

2. For each prompt draw a graph without self-loops or explain why no such graph exists:

(a) 7 vertices of degree 2, 2 vertices of degree 3 and 2 vertices of degree 1.

(b) 5 vertices of degree 5 and 1 of degree 1.

(c) 3 vertices of degree 3 and 10 of degree 4.

Solution:

(a) Can be drawn as a cycle of 9 vertices with two vertices stick out the sides of it.

(b) Can't be drawn. Let v be one of the vertices with degree 5, then v must be connected to the four other vertices with degree 5 and the vertex of degree 1. When we go to draw our next vertex of degree 5 we no longer have enough vertices to connect it to as the vertex of degree 1 is already called for by v.

(c) Can't be drawn. This is by the handshake lemma: the sum of degrees of all vertices across a graph is equal to 2 times the number of edges and is even. If $d_G(v)$ is the degree of a vertex then we write this lemma as:

$$\sum d_G(v) = 2|E(G)|$$

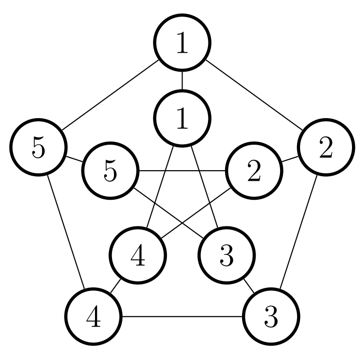
.

3. Prove that every loopless and connected graph with at least 2 vertices contains 2 vertices of the same degree.

Solution: Assume each vertex has a distinct degree. Then in a graph of size n we have a vertex with degree $n-1$, one with $n-2$, one with $n-3$... until we reach a vertex with degree 0 which can't be true as the graph is connected.

II. Intermediate Problems

4. Below is the Petersen Graph.



Is it possible to decompose the graph into 3 paths of 5 vertices?

Solution: No. Think about it this way: each vertex in the graph has degree three so in a decomposition into paths each vertex will either be in 2 paths, P_1 and P_2 where there are an end point in one and a midpoint in the other or they are in three paths as end points of the paths. Therefore each vertex is an end point in at least one path but we are only decomposing into 3 paths (6 endpoints) and we have 10 vertices.

5. A planar graph G is a graph that can be drawn in the plane with no edges crossing except at vertices. Let G have v vertices, e edges, and f faces. Note that the outer "face" is counted in f. Prove that for every connected planar graph Euler's formula holds: $v + f - e = 2$.

Solution: Use induction.

6. Prove that if a planar graph has n vertices and more than $3n/2$ edges, the graph must have at least $f/10$ faces with the same number of sides.

Solution: Use Euler's formula

III. Advanced Problems

7. A complete graph K_n is a graph where each vertex shares an edge with every other vertex in the graph. Let each edge, e, in such a complete graph be given an integer label, $x(e)$ such that no two incident edges share a label (ie. no edges that share a vertex have the same label). Show there is a trail $e_1, e_2, \ldots e_{n+1}$ of $n-1$ edges where the labels in the trail are increasing ($x(e_i) < x(e_j)$ if $i < j$).

Solution: Let $p_G(v)$ denote the longest trail ending at $v \in V(G)$ st. the labellings of edges in $p_G(v)$ are increasing. Let $|p_G(v)|$ denote the number of edges in the trail $p_G(v)$ then $\sum |p_G(v)|$ is the sum across all vertices in G of the length (in edges) of the longest trail ending at each vertex.
Let our graph be K_n with the labelling as described in the problem and remove the largest labelled edge (over the whole graph), e, that connects vertices v_1 and v_2 from the graph. Let $x = p_{K_n - e}(v_1)$ and $y = p_{K_n - e}(v_2)$ in this graph. Then returning e to the graph we have three cases:

i $|x| < |y|$ which implies that (since the label of e is larger than any other incident on v_1 or v_2) the trail made by extending y with e is longer than x. Therefore when we return e to the graph we have $p_{K_n}(v_1)$ equal to y with the addition of e. Therefore we've increased the sum of all $|p_{K_n}(v)|$ by at least two (as $|y| - |x| \geq 1$)

ii $|y| > |x|$ is exactly the same as case one with the vertices swapped.

iii $|y| = |x|$ then returning e to the graph we can extend both of the trails ending at v_1 and v_2 by taking the others trail and tacking e onto the end ie. $p_{K_n}(v_1) = y + \{ev_1\}$ and $p_{K_n}(v_2) = x + \{ev_2\}$. Thus increasing the size of both trails by 1 (and the overall sum by 2).

We remove all edges from K_n and return the edges 1 by 1 from smallest label to greatest label. For each edge, e, that we return one of the above three cases above occurs between the end points of e. Therefore each time we return an edge we add at least 2 to $\sum |p_{K_n}(v)|$. There are $\frac{n(n-1)}{2}$ edges in a complete graph therefore $\sum |p_{K_n}(v)| \geq n(n-1)$. This means that the average $|p_{K_n}(v)|$ for vertices in K_n is greater than or equal to $\frac{n(n-1)}{n} = n - 1$ and thus we must have at least one vertex with $|p_{K_n}(v)| \geq n - 1$ implying it has a trail of length greater than or equal to $n - 1$ with all edge labels increasing.