UBC Math Circle 2019 Problem Set 4

Problems will be ordered roughly in increasing difficulty

1. The numbers $1, 2, ...2k$ are given. You may remove two numbers, x, y, and replace them with $x + xy + y$. After $2k - 1$ iterations, there is one element left. What are the possibilities for this element?

Solution: First, a simple example. Given x, y, z, we have, after 1 step, $xy + x + y, z$, and after the second step, $xyz + xz + yz + xy + x + y + z$. Thus, by induction, we see that we are summing all possible products of subsets of the given numbers. This is $(2k + 1)! - 1$. Note that having an even number was a red herring.

2. The integers $1, 2, ...n$ are written down in that order. At each step, you may swap any two numbers. Prove that you can never return to the starting arrangement after an odd number of steps.

Solution: Let $P = x_1, x_2, ...x_n$ be any permutation of the integers $1, 2, ...n$. Let $u_P(k)$ be the number of $x_i > x_k$, for $1 \leq i < k$. Let $U(P) = \sum_{k=1}^{n} u_P(k)$. We have that $U(P)$ measures how unordered the permuation is. We note that for the permutation $I = 1, 2, ...n$, that $U(I) = 0$. Now consider what happens if P and P' differ by exactly 1 swap. Let x_k and x_m be the swapped pair in question, so we have $P = x_1, ...x_k, ...x_m, ...x_n$ and $P' = x_1, ...x_m, ...x_k, ...x_n$. We want to consider $U(P) - U(P')$. Note that $u_P(i) = u_{P'}(i)$ for $i < k$ and $i > m$. Now consider $u_P(i) - u_{P'}(i)$ for $k < i < m$. We have a few cases:

Case 1: $x_k < x_i > x_m$ or $x_k > x_i < x_m$. Then $u_P(i) - u_{P'}(i) = 0$.
Case 2: $x_k < x_i < x_m$. Then $u_P(i) - u_{P'}(i) = -1 - 1 = -2$.
Case 3: $x_k > x_i > x_m$. Then $u_P(i) - u_{P'}(i) = 1 - (-1) = 2$.

Finally, we have to consider $u_P(k) - u_{P'}(k)$ and $u_P(m) - u_{P'}(m)$, (Being careful because in P', the k-th position is actually occupied by x_m and vice versa). We now have 2 cases:

Case 1: $x_k < x_m$. Then $u_P(k) - u_{P'}(k) + u_P(m) - u_{P'}(m) = -1$.
Case 2: $x_k > x_m$. Then $u_P(k) - u_{P'}(k) + u_P(m) - u_{P'}(m) = 1$.

Thus, in the end, $U(P) - U(P')$ is an odd number $(-3, -1, 1, 3)$, and we can't have an odd sum of odd number be zero, so we can't get back to the original permutation after an odd number of swaps.

3. Start with a sequence $a_1, a_2, ..., a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\text{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop, and the final sequence is determined entirely by the initial sequence, ie it is independent of the choices made.

Solution: We first prove that the process stops. Note first that the product $a_1 \cdots a_n$ remains constant, because $a_j a_k = gcd(a_j, a_k) lcm(a_j, a_k)$. Moreover, the last number in the sequence can never decrease, because it is always replaced by its least common multiple with another number. Since it is bounded above (by the product of all of the numbers), the last number must eventually reach its maximum value, after which it remains constant throughout. After this happens, the next-to-last number will never decrease, so it eventually becomes constant, and so on. After finitely many steps, all of the numbers will achieve their final values, so no more steps will be possible. This only happens when a_j divides a_k for all pairs $j < k$.

We next check that there is only one possible final sequence. For p a prime and m a nonnegative integer, we claim that the number of integers in the list divisible by p^m never changes. To see this, suppose we replace a_j, a_k by $gcd(a_j, a_k)$ and $lcm(a_j, a_k)$. If neither of a_j, a_k is divisible by p^m, then neither of $gcd(a_j, a_k)$ and $lcm(a_j, a_k)$ is either. If exactly one a_j, a_k is divisible by p^m, then $lcm(a_j, a_k)$ is divisible by p^m but $gcd(a_j, a_k)$ is not. If both of a_j, a_k are divisible by p^m, then $gcd(a_j, a_k)$ and $lcm(a_j, a_k)$ are as well.

If we started out with exactly h numbers not divisible by p^m, then in the final sequence $a'_1, ...a'_n$, the numbers $a'_{h+1}, ...a'_n$ are divisible by p^m while the numbers $a'_1, ...a'_h$ are not. Repeating this argument for each pair (p, m) such that p^m divides the initial product $a_1, \cdots, a_n$, we can determine the exact prime factorization of each of $a'_1, ...a'_n$. This proves that the final sequence is unique.

4. Suppose we have an infinite grid, with n points on the grid coloured blue. Iteratively colour points on the grid if they have two adjacent points coloured in (adjacent as in horizonally or vertically, diagonals do not count). Prove that this process terminates, and give an upper bound on the final number of coloured points. Determine which starting arrangements can achieve this bound.

Solution: This problem is harder than initially thought. We don't know of a solution.

5. Show that every convex polyhedron has at least two faces with the same number of sides.

Solution: Choose the largest face, and suppose that it has n sides. Then, the number of sides that each of its n neighbours has is in the set $S = 3, ..., n$ sides. However, $n > |S|$, so pigeonhole gives the desired result.

6. Let P be a convex polygon. Show that there exists consecutive vertices A, B, and C such that the circumcircle of $\triangle ABC$ covers P.

Solution: Among the finitely many circles through three vertices of the n-gon, there is a maximal circle.

Now we split the problem into 2 parts:

(a) the maximal circle covers the n-gon.

(b) the maximal circle passes through three consecutive vertices.

We prove (a) indirectly. Suppose the point A' lies outside the maximal circle about triangle ABC where A, B, C are denoted such that A, B, C, A' are vertices of a convex quadrilateral. Then the circumcircle of triangle A'BC has a larger radius then that of triangle ABC. Contradiction.

We also prove (b) indirectly. Let A, B, C be vertices on the maximal circle, and let A' lie between B and C and not on the maximal circle. Because of (a), it lies inside that circle, but then the circle about triangle A'BC is larger than the maximal circumcircle. Contradiction.

7. Show that any convex polygon of area 1 is contained in a rectangle of area 2.

Solution: Let P be a convex polygon with area 1 and let AB be a diameter of P. Let l and m be the lines perpendicular to AB through A and B respectively. Let n and o be lines perpendicular to l and m such that just meet P (For example, to construct n, draw a line perpendicular to l and m away from K on the right. Move this line towards P until it first touches P). Observe that n and o must each intersect at least one vertex of P each. Call these vertices C and D, respectively. We have that the four lines l, m, n, o form a rectangle that bounds P, which we shall call R. The diameter AB splits R into two smaller rectangles, which we can call S and T, containing C and D, respectively. Consider the triangle ABC, which lies entirely in S, and likewise, ABD lies entirely in D. It is now clear that the area of ABC is half the area of S, and similarly for ABD. The quadrilateral $ACBD$, has area equal to the sum of the areas of ABD and ABC, which by the above, is half the area of R. However, $ACBD$ is contained entirely in P because P is convex, so $ACBD$ has area at most 1. Hence, R has area at most 2.