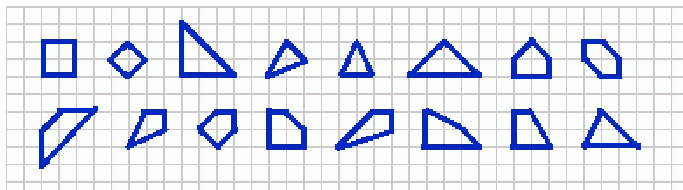
UBC Math Circle 2020 Problem Set 3

Problems will be ordered roughly in increasing difficulty

1. We say that two lattice polygons are equivalent if there is a rotation, translation, shear transformation, or combinations of these. Find all 16 convex lattice polygons with exactly one point in the interior, up to equivalence. You do not need to prove that there are exactly 16.

Solution:



2. Victor is interested in generalizing the following observation: 2 chopsticks can hold 1 food item, while 1 chopstick is completely useless.

   Victor models the maximal number of food items that can be held by n chopsticks to be the maximal number of edges in a planar graph with n vertices (each vertex represents a chopstick, and each edge represents a food item that can be picked up by that pair of chopsticks).

   Find, with proof, a closed formula for the maximal number of food items that can be picked up by n chopsticks, according to this model.

Solution: The maximum is $3(n-2)$ for every $n > 2$ (the maxima of $0, 0, 1$ for $n = 0, 1, 2$ are clear). The upper bound follows from Euler's relation $F - E + V = 2$ (note a maximum configuration must be connected) upon setting $V = n$ and using the inequality $E \geq \frac{3}{2}F$ (each edge is on the boundary of at most 2 faces, and each face must have a cycle of length ≥ 3 on its boundary, except in the case where there is only one face, in which case the boundary includes all edges). This inequality is sharp iff every face is a triangle. So $E = 3n - 6$ can be attained inductively: start from a triangle for $n = 3$ and then to get from n to $n + 1$ put a new vertex in some triangle and connect it to the triangle's three vertices to split the original triangle into three.

Try this at home (if you wish)! It is difficult enough to try picking up 3 food items when 3 chopsticks. Trying to attain the maximum for larger n is near impossible in a physical setting.

3. Three lattice points A, B, C are chosen in a plane. Prove that if $\triangle ABC$ is acute, then at least one lattice point is inside or on its sides.

Solution: Suppose for contradiction that $\triangle ABC$ is acute and no lattice point is inside or on its sides. From Pick's formula, we get $S_{\triangle ABC} = \frac{1}{2}$. Suppose $\angle C$ is the largest angle in the triangle. Then $60° \le \angle C < 90°$. So $\sin C \ge \frac{\sqrt{3}}{2}$.

$\Rightarrow \frac{1}{2} = S_{\triangle ABC} = \frac{1}{2} ab \sin C \ge \frac{\sqrt{3}}{4} ab$

$\Rightarrow a^2 b^2 \le \frac{4}{3}$.

Since A, B, C are lattice points, a^2 and b^2 are positive integers. From the inequality above, the only possibility is that $a = b = 1$. Then $\sin C = 1$, i.e. $\angle C = 90°$, which contradicts that $\triangle ABC$ is acute.

4. Take any convex n-gon. Select any m points inside of it. Cut the polygon into nonintersecting triangles whose vertices are these $m + n$ points. How many triangles do you get in terms of m and n?

Solution: Let x be the number of triangles formed. We can compute the sum S of their angles in two ways. On the one hand, $S = 180°x$. On the other hand, $S = 360°m + 180°(n - 2)$. The first term is the sum of the angles of all m interior points. The second term is the sum of the angles of an n-gon. By equating the right sides of the two equations, we get $x = 2m + n - 2$. Alternatively, we can use induction, or use Eulers formula $f + v = e + 2$