UBC Math Circle 2020 Problem Set 7

Problems will be ordered roughly in increasing difficulty

1. Find the smallest n such that given any set of n integers there is always a pair of integers whose difference is divisible by 2020.

Solution: $n = 2020$ does not work, consider the set $\{0, 1, ...2019\}$. The differences in this set are bounded by 2019 in absolute value, yet none of them are 0 because we don't have any duplicates. We show $n = 2021$ works. Consider the possible remainders modulo 2020. There are 2020 such possible remainders, and 2021 elements, so there has to be a remainder that is repeated. Then pick the two elements with this remainder, their difference has remainder 0.

2. A course has 7 elective topics, and students must complete exactly 3 of them in order to pass the course. If 200 students passed the course, show that at least 6 of them must have completed the same electives as each other.

Solution: There are $\binom{7}{3} = 35$ possible choices of 3 completed courses. In the worst case scenario, every passing student passed only 3 courses. By pigeonhole, we must have at least 6 students who pass the same 3 electives.

3. Prove that you can't arrange 100 points inside a 13×18 rectangle so that the distance between any two points is at least 2.

Solution: We wish to fit a circle of radius 1 around every point. The area of such a circle would be π, but not all of it will necessarily fit inside the rectangle. Thus, we can enlarge the rectangle by 1 unit, and assert that all the points have to lie inside the original 13×18 rectangle. Then these circles will lie inside a 15×20 rectangle. The total area we have to work with is 300, yet the area of 100 circles would be $100\pi > 300$, so there would have to be a pair of overlapping circles, which means there are points closer than distance 2 apart.

4. Consider an n-gon with vertices $x_1, \ldots, x_n$. Assign the numbers $1, ...n$ randomly to each vertex. Let $m \leq n$. Show that there exists m consecutive vertices $x_k, x_{k+1}, \ldots, x_{k+m-1}$ (where $k + m - 1$ is taken modulo n) such that the sum of the values assigned to $x_k, \ldots, x_{k+m-1}$ is at least $\frac{m(n+1)}{2}$.

Solution: Let S_i be the sum of m consecutive vertices, starting with x_i. For example, $S_1 = x_1 + x_2 + \cdots + x_m$, and $S_n = x_n + x_1 + \cdots + x_{m-1}$. We have that $\sum_{i=1}^n S_i = m(\sum_{i=1}^n x_i) = m(\frac{n(n+1)}{2})$. By pigeonhole, this has to be divided among n holes, the S_i, so there must be some S_i with value at least $\frac{m(n+1)}{2}$.

5. Find the smallest n such that given any n points in the plane, there exists a 3-tuple of points, (A, B, C) such that $\angle ABC \leq 30°$.

Solution: In general, to obtain an angle of $\frac{180°}{n}$, one needs at least n points, and n points suffices. To show that $n-1$ points is not enough, consider the regular $(n-1)$-gon. To show that n points is always enough, consider the convex hull of the n points. This is at most an n-gon, so there is an angle with size at most $\frac{180(n-2)}{n}$. If the convex hull has less than n points, this angle would be $180 - \frac{360}{m} < 180 - \frac{360}{n}$ for $m < n$ points on the convex hull.

Now we can draw a line from the $n-3$ remaining points to the vertex with angle $\leq 180 - \frac{360}{n}$. This divides it into $n-2$ smaller angles, so there is one with at most $\frac{180}{n}$.

6. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \ldots, x_n$, not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ of elements of $\{-1, 0, 1\}$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$.

Solution: The statement holds for $n = 9$ by choosing $\{2^0, 2, 2^2, 2^3, \ldots, 2^8\}$. In this case

$$|\epsilon_1 2^0 + \cdots + \epsilon_9 2^8| \leq 1 + 2 + \cdots + 2^8 < 9^3$$

However, if $n = 10$, then $2^{10} > 10^3$, so by the pigeonhole principle, there are two subsets A and B of $x_1, ..., x_{10}$ whose sums are congruent modulo 10^3. Let $\epsilon_i = 1$ if x_i occurs in A but not in B, -1 if x_i occurs in B but not in A, and 0 otherwise; then $\sum \epsilon_i x_i$ is divisible by n^3.