UBC Math Circle 2020 Problem Set 8

Problems will be ordered roughly in increasing difficulty

1.
   - Consider tic-tac-toe on a torus. This can be imagined by having the sides wrap around like in space invaders. For example :

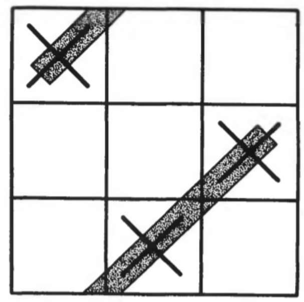
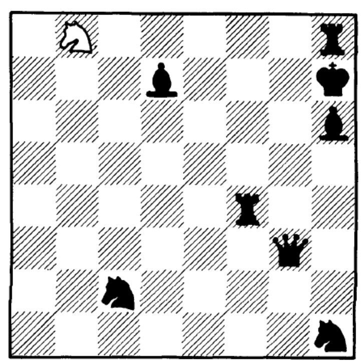
     

     Figure 2.2 These Xs are three-in-a-row if the board is imagined to represent a torus.

     From The Shape of Space by Jeffrey Weeks

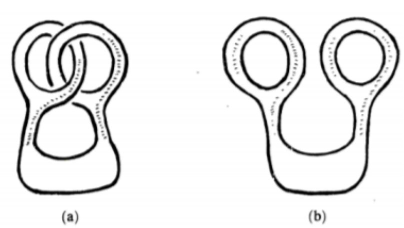
     Play tic-tac-torus with several different people. Try to find a winning strategy.

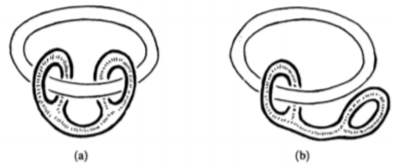
   - Consider chess on a torus. Which pieces can the white knight capture (in the figure below)?

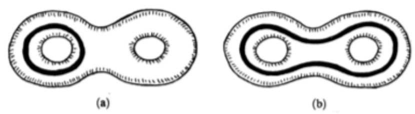
     

   - Consider again chess on a torus. Determine whether or not (with proof) a knight and a bishop can simultaneously threaten each other.

2. Show that figure (a) can be continuously deformed into figure (b), for each set of figures below. Here, a continuous deformation allows stretching, squishing, and moving about, but no cutting, tearing, or poking holes in objects, which we imagine are made of a material like play-doh.





3. Let S^n denote the n-dimensional sphere, ie the set of points $x_0^2 + ... + x_n^2 = 1$ in $\mathbb{R}^{n+1}$ (Note that the sphere is hollow, much like the regular sphere and circle). Let B^n denote the n-dimensional ball, ie the set of points $x_0^2 + ... + x_{n-1}^2 \leq 1$ in $\mathbb{R}^n$ (This is the solid object). We admit the following theorem, without proof.

   Theorem 1 (Borsuk-Ulam)**.** *For every continuous function $f : S^n \to \mathbb{R}^n$, there exists a point $x = (x_0, ..., x_n) \in S^n$ such that $f(x) = f(-x)$.*

   There is also the equivalent formulation, which will be useful (you do not need to prove that it is equivalent):

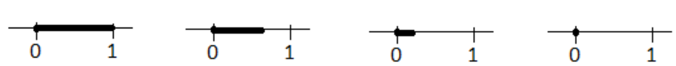
   Theorem 2 (Borsuk-Ulam Version 2)**.** *There is no continuous function $f : B^n \to S^{n-1}$ such that $f(-x) = -f(x)$ on the boundary of the ball, which is S^{n-1}.*

   Use the Borsuk-Ulam theorem to prove the following Brouwer fixed point theorem:

   Theorem 3 (Brouwer fixed point)**.** *Every continuous function $f : B^n \to B^n$ has a fixed point, ie there is a point $x \in B^n$ with $f(x) = x$.*

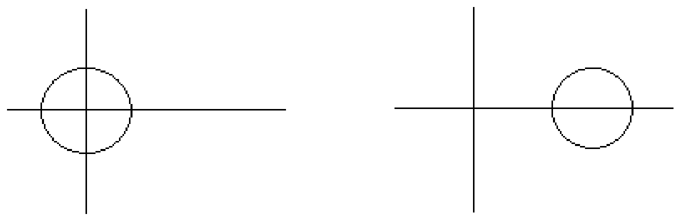
   Hint: Suppose for contradiction there was a continuous function that had no fixed point. Thus the points x and $f(x)$ are always distinct. Use these points to define a third point, call it $g(x)$. Show $g(x)$ contradicts Borsuk-Ulam. Try visualizing for $n = 3$ first.

The next questions deal with the concept of *homotopy*. A homotopy between two objects in a space is a continuous deformation that starts at one object, and ends at the other, staying within the space the entire time. Question 1 gives examples of homotopies of objects in $\mathbb{R}^3$ (In fact, they are examples of something stronger, known as ambient isotopy). Another example is there is a homotopy between the line segment $[0, 1]$ and the point $\{0\}$ in $\mathbb{R}$, as illustrated below.

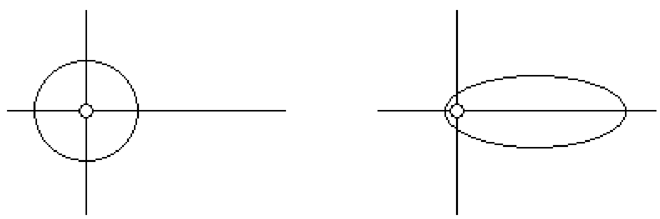


Continuously shrinking the interval [0,1] to the point {0}

When we can get from one object to another via a homotopy, we call the two objects homotopic. Thus to prove that two objects are homotopic, all one needs to do is find a homotopy. However, to prove that two objects are not homotopic, one usually requires some heavy mathematical tools. Thus, for the sake of these problems, an intuitive explanation for being non-homotopic suffices. For example, the circle $x^2 + y^2 = 1$ is homotopic to the circle $(x-2)^2 + y^2 = 1$ in all of $\mathbb{R}^2$, but if we ask the same question in $\mathbb{R}^2 \setminus \{(0,0)\}$, it is no longer true because the circle gets "caught" on the missing point.

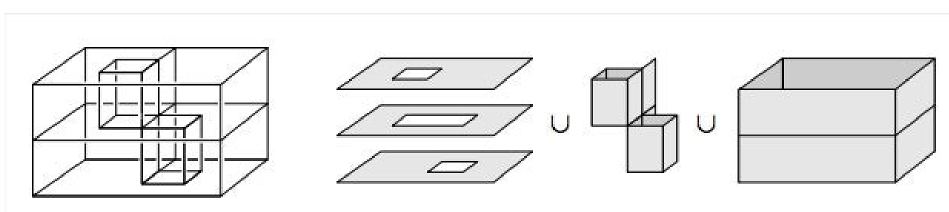


The circle can simply be translated to the right



The circle gets caught on the missing point

4. Show that there is a homotopy between the following space and a point (the picture can be hard to see clearly). Pay attention to the walls and ceiling of the hollow region.



5. Of particular interest is the homotopy of loops - a continuous path in the space that starts and ends at the same point. The example for homotopy shows some basic loops in $\mathbb{R}^2$ and $\mathbb{R}^2 \setminus \{(0,0)\}$. Loops are allowed to have self intersections. Consider again $\mathbb{R}^2 \setminus \{(0,0)\}$. Determine all the different loops in this space, up to homotopy. Here two loops are the same if there is a homotopy between them.
Bonus: Consider the same problem, except in a solid torus. Do it again but for a (hollow) torus.