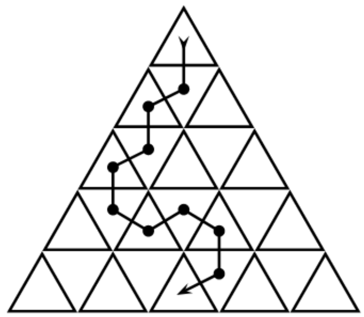
UBC Math Circle 2021 Problem Set 10

1. A triangular grid is obtained by tiling an equilateral triangle of side length n by n^2 equilateral triangles of side length 1. Determine the number of parallelograms bounded by line segments of the grid. (See the below picture, but ignore the path drawn.)



2. A 6×6 square is tiled with 2×1 dominoes. Prove that it's possible to cut the board into two smaller rectangles with a straight line which doesn't pass through any of the dominoes.

3. Victor needs to write an essay on how much he loves the planet for his English class. Victor is not a very creative writer, so he writes his essay by repeatedly choosing a character among the 26 letters of the alphabet uniformly at random. Is it more likely for the word "HEART" or "EARTH" to appear first in Victor's essay?

4. Prove that a set A is infinite if and only if there exists a strict subset $B \subsetneq A$ such that there exists a bijection (a one-to-one and onto function) between A and B.

5. Find all pairs of integers (m, n) such that there exists a pair of integers (x, y) so that every element of $\mathbb{Z}^2$ may be written as $a(m, n) + b(x, y)$ for some integers a, b.

6. This question roughly asks for an algorithm such that given a list of nonnegative integers $(x_1, \ldots, x_{2n+1})$, where $n \geq 1$, in which exactly one entry appears an odd number of times, the algorithm returns the "odd one out".

   For example, given $(1, 4, 1, 3, 4, 3, 3, 4, 4)$, the algorithm should return 3.

   (a) Do there exist functions $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ and $f_k : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ for $k \in \mathbb{N}$ such that for any list $(x_1, \ldots, x_{2n+1})$ satisfying the necessary hypotheses described in the problem statement, the expression

   $$g(f_{2n}(f_{2n-1}(\ldots f_2(f_1(x_1, x_2), x_3) \ldots, x_{2n}), x_{2n+1}))$$

   always returns the "odd one out"?

(b) Given a positive constant c and positive integer n, let A_c denote the set $\{m \in \mathbb{Z} : 0 \leq m \leq cn\}$.

Fix a positive integer n. Does there necessarily exist a constant $c > 0$ and functions $g : A_c \to A_c$ and $f_k : A_c \times A_c \to A_c$ for $1 \leq k \leq 2n$ such that for any list $(y_1, \ldots, y_{2n+1})$ satisfying the necessary hypotheses described in the problem statement, there exists some rearrangement $(x_1, \ldots, x_{2n+1})$ of the list such that the expression

$$g(f_{2n}(f_{2n-1}(\ldots f_2(f_1(x_1, x_2), x_3) \ldots, x_{2n}), x_{2n+1}))$$

always returns the "odd one out"?

(c) Fix a positive integer n. Does there necessarily exist a constant $c > 0$ and functions $g : A_c \to A_c$ and $f_k : A_c \times A_c \to A_c$ for $1 \leq k \leq 2n$ such that for any list $(x_1, \ldots, x_{2n+1})$ satisfying the necessary hypotheses described in the problem statement, the expression

$$g(f_{2n}(f_{2n-1}(\ldots f_2(f_1(x_1, x_2), x_3) \ldots, x_{2n}), x_{2n+1}))$$

always returns the "odd one out"?