UBC Math Circle 2021 Problem Set 2

1. (a) Let *Stirling numbers of the second kind* be denoted by $\left\{ n \atop k \right\}$, which counts the number of ways to partition a set of n elements into k nonempty subsets. Prove that they satisfy the recurrence

$$\left\{ n+1 \atop k+1 \right\} = (k+1)\left\{ n \atop k+1 \right\} + \left\{ n \atop k \right\}.$$

> **Solution:** The number of ways to partition the set $[n+1] = \{i \in \mathbb{N} : i \leq n+1\}$ into $k+1$ nonempty subsets is given by adding the number of partitions where $n+1$ is in a singleton part, and the number of partitions where $n+1$ is not in a singleton part.
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> The former is given by $\left\{ n \atop k \right\}$, since it counts the number of ways to partition $[n]$ into k nonempty subsets, and the latter is given by $(k+1)\left\{ n \atop k+1 \right\}$, by counting the number of ways to partition $[n]$ into $k+1$ nonempty subsets, and then multiplying by $k+1$, the number of ways to insert the element $n+1$ into one of those parts.

(b) Prove for all n and all $x \in \mathbb{R}$ that

$$x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} x(x-1)...(x-k+1).$$

You may or may not want to use part (a).

> **Solution:** We first prove the identity for the special case where $x \in \mathbb{N}$.
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> The number of elements in $[x]^n$ with k distinct entries is $\left\{ n \atop k \right\} x(x-1)...(x-k+1)$, since for each partition of $[x]$ into k parts, there are $x(x-1)...(x-k+1)$ choices of an element $(a_1, ..., a_n) \in [x]^n$ such that $a_i = a_j$ iff i and j are in the same set part.
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> Summing over $0 \leq k \leq n$ shows the identity for $x \in \mathbb{N}$. Now both sides of the equation are real polynomials in x that agree on infinitely many points, so by Lagrange interpolation they must be the same polynomial. Thus the identity holds for all $x \in \mathbb{R}$.

2. (CMO 2015) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $(n-1)^2 < f(n)f(f(n)) < n^2 + n$ for every $n \in \mathbb{N}$.

Solution: `https://www2.cms.math.ca/Competitions/CMO/archive/sol2015.pdf`

3. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(a)) = a + 1$ for all $a \in \mathbb{Z}$.

Solution: There are none. Since the map $a \mapsto a + 1$ is a bijection $\mathbb{Z} \to \mathbb{Z}$, any such $f : \mathbb{Z} \to \mathbb{Z}$ must also be a bijection.

Given any bijection $g : \mathbb{Z} \to \mathbb{Z}$, define the equivalence relation $\sim_g$ on $\mathbb{Z}$ such that $a \sim_g b$ if and only if $g^k(a) = b$ for some $k \in \mathbb{Z}$.

Note that the (possibly infinite) number of equivalence classes of $\sim_{f^2}$ is always $\geq$ the number of equivalence classes of $\sim_f$.

Since $\sim_{a \mapsto a+1}$ has only one equivalence class, any f satisfying the problem statement must have $\sim_f$ have only one equivalence class. Then f is an infinite cycle on $\mathbb{Z}$ (it must be of the form $f(g(a)) = g(a + 1)$ where $g : \mathbb{Z} \to \mathbb{Z}$ is some bijection), so $\sim_{f^2}$ has two equivalence classes ($\{g(a) : a \text{ even}\}$ and $\{g(a) : a \text{ odd}\}$), so we can't have $f(f(a)) = a + 1$ for all $a \in \mathbb{Z}$.

4. (a) Let $a, b, c \in \mathbb{Z}$ be coprime to 13. Show that $a^3 + b^3 + c^3$ is also coprime to 13.

Solution: Suppose, for a contradiction, that there exist $a, b, c \in (\mathbb{Z}/13\mathbb{Z})^\times$ such that $a^3 + b^3 + c^3 = 0$ (equations will be understood to be in the field $\mathbb{Z}/13\mathbb{Z}$).

Multiplying both sides by c^{-3} and rearranging, we see that $(ac^{-1})^3 + 1 = (-bc^{-1})^3$. So there exist $x, y \in (\mathbb{Z}/13\mathbb{Z})^\times$ such that $x^3 + 1 = y^3$. The set of nonzero cubes in $\mathbb{Z}/13\mathbb{Z}$ is $\{-8, -1, 1, 8\}$, so this is impossible.

(b) Does there exist a configuration of 13 people such that in every group of 3 there is a pair of mutual strangers, and in every group of 5 there is a pair of mutual friends?

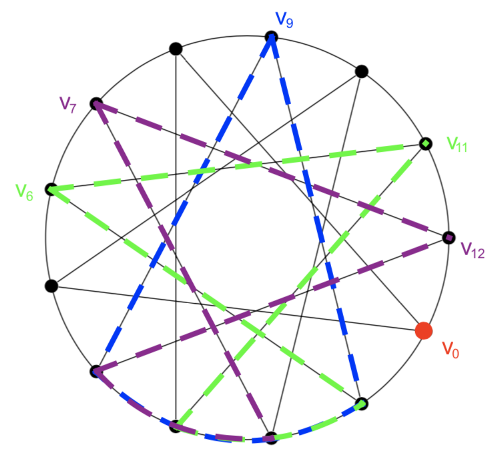
Solution: Yes. Label the 13 people $v_0, \ldots, v_{12}$ by the elements of $\mathbb{Z}/13\mathbb{Z}$. Suppose their friendship configuration is such that v_i and v_j are friends if and only if $i - j$ is a nonzero cube in $\mathbb{Z}/13\mathbb{Z}$. Call their friendship graph G.

By part (a), there don't exist v_i, v_j, v_k distinct that are all friends, or else $(j - i) + (k - j) + (i - k)$ would be a sum of three nonzero cubes summing to zero in $\mathbb{Z}/13\mathbb{Z}$.

Next we show that there doesn't exist a set of 5 people with no pair knowing each other. This can be done by casework. For completeness, we give one possible argument below.

Suppose, by way of contradiction, that there does exist a K_5 in $\overline{G}$. Then, there exists an independent set of 5 (distinct) vertices $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}$ in G. By symmetry of G, the vertices $v_{i_1+1}, v_{i_2+1}, v_{i_3+1}, v_{i_4+1}, v_{i_5+1}$ also form an independent set in G. Note we never have $v_{i_j} = v_{i_k+1}$ for any $1 \leq j, k \leq 5$, or else $v_{i_k}v_{i_j} = v_{i_k}v_{i_k+1}$ is an edge in an independent set. So these are 10 distinct vertices with the property that the induced graph on these vertices is bipartite.

Let the set W be the 3 other vertices of G. For each vertex $w \in W$, let $c(w)$ be the number of edges from w to the next element of W when traversing the outer cycle clockwise. Since $\sum_{w \in W} c(w) = 13 > 3 \cdot 4$, one of the vertices $w \in W$ satisfies $c(w) \geq 5$. By symmetry of G, we may, without loss of generality, let it be v_0.



Since $v_0 \in W$ satisfies $c(v_0) \geq 5$, we know $v_1, v_2, v_3, v_4 \in V(G) - W$. Note $v_1v_2v_3v_{11}v_6v_1$ (green), $v_1v_2v_3v_4v_9v_1$ (blue), and $v_2v_3v_4v_{12}v_7v_2$ (purple) are cycles of length 5 contained in G. By the pigeonhole principle, the remaining 2 vertices of W must have empty intersection with at least one of the sets $\{v_6, v_{11}\}$, $\{v_9\}$, $\{v_7, v_{12}\}$. So there is a cycle of length 5 contained in G that only uses the vertices of $V(G) - W$. But then the induced graph on the vertices $V(G) - W$ cannot be bipartite, contradiction.

So there cannot exist a K_5 in $\overline{G}$.

Thus this configuration of 13 people has the property that in every group of 3 there is a pair of mutual strangers and in every group of 5 there is a pair of mutual friends.

5. There are n markers, each with one side white and the other side black. In the beginning, these n markers are aligned in a row so that their white sides are all up. In each step, if possible, we choose a marker whose white side is up (but not one of the outermost markers), remove it, and reverse the closest marker to the left of it and also reverse the closest marker to the right of it. Prove that if $n \equiv 1 \pmod 3$ its impossible to reach a state with only two markers remaining.

Solution: Let ω be a primitive cube root of unity, say $\omega = e^{2\pi i/3}$. Consider the bijections from the set $\{1, \omega, \omega^2\}$ to itself. Let b be the bijection sending $x \mapsto \frac{1}{x}$ and w be the bijection sending $x \mapsto \omega x$.

Given a sequence of black and white markers, associate with it a composition of these b and w bijections. For example, if the markers are black, white, then black, associate with it the bijection $b \circ w \circ b$.

Then check that $w \circ w \circ w = b \circ b$, $w \circ w \circ b = b \circ w$, $b \circ w \circ w = w \circ b$, and $b \circ w \circ b = w \circ w$. Therefore, the associated bijection with a sequence of markers remains invariant when we apply any legal move in each step.

So when we start with $n \equiv 1 \pmod 3$ white markers, the associated bijection is $\underbrace{w \circ \cdots \circ w}_{n=3k+1 \text{ times}} = w$. Checking that none of $w \circ w$, $w \circ b$, $b \circ w$, $b \circ b$ are equal to w completes the proof.

Remark: The set of bijections from $\{1, \omega, \omega^2\}$ to itself is also known as the *symmetric group on the set of* 3 *elements* or the *dihedral group of order* 6 in abstract algebra.

6. Let $x, y \in \mathbb{R}$ satisfy $x + y = 1$. Show for all $m, n \in \mathbb{N}$ that

$$x^{n+1} \sum_{j=0}^{m} \binom{n+j}{j} y^j + y^{m+1} \sum_{j=0}^{n} \binom{m+j}{j} x^j = 1.$$

Solution: First, we consider the case when x and y are both positive. Consider a biased coin that gives heads with probability x and tails with probability y. We can do this because $x + y = 1$. Suppose we continuously flip this biased coin. The probability that we flip $n + 1$ heads and j tails with the last flip being heads is

$$\binom{n+j}{j} x^{n+1} y^j,$$

because we should choose j of the first $n + j$ flips to be tails. This means that the probability of flipping $n + 1$ heads before flipping $m + 1$ tails is

$$x^{n+1} \sum_{j=0}^{m} \binom{n+j}{j} y^j.$$

Similarly, the probability of flipping $m + 1$ tails before flipping $n + 1$ heads is

$$y^{m+1} \sum_{j=0}^{n} \binom{m+j}{j} x^j.$$

Since the event of flipping $n+1$ heads before $m+1$ tails and the event of flipping $m+1$ tails before $n+1$ heads are disjoint, and exactly one of the events must happen, we get the desired equality when x and y are both positive:

$$x^{n+1}\sum_{j=0}^{m}\binom{n+j}{j}y^j+y^{m+1}\sum_{j=0}^{n}\binom{m+j}{j}x^j=1.$$

To finish the proof, we show that x and y need not be positive. Since we may write $y=1-x$, both sides of the equality are polynomials of one variable. Since they agree at infinitely many points, they must be the same for all points. This is because the polynomial

$$x^{n+1}\sum_{j=0}^{m}\binom{n+j}{j}y^j+y^{m+1}\sum_{j=0}^{n}\binom{m+j}{j}x^j-1$$

has more zeros than its degree, so it must be the zero polynomial.