UBC Math Circle 2021 Problem Set 3

1. Is it possible to tile a $6 \times 6 \times 18$ box with $1 \times 2 \times 4$ bricks?

2. Let $[n]$ denote the set $\{1, 2, 3, \ldots, n-1, n\}$

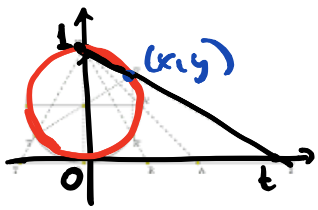
   (a) Show that for every k, there is a subset $S \subset [2k]$ with k elements such that for every choice of distinct $a, b \in S$, $a \nmid b$ (a does not divide b).

   (b) Show that for every k, and any subset $S \subset [2k]$ with at least $k+1$ elements, we can find some distinct $a, b \in S$ such that $a \mid b$ (a divides b).

   Similar facts are true for $[2k+1]$. Feel free to think about what small changes need to be made to adapt the problem to $[2k+1]$.

Problems on Projective Geometry. February 1, 2021.

1. Find a formula for the coordinates (x, y) of the point on the circle that corresponds to the point t on the line under the stereographic projection:

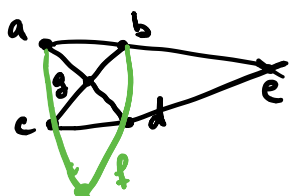


2. (a) Find how many solutions does the equation $x^2 + y^2 = 1$ have when $x, y \in \mathbb{F}_3 = \{0, 1, 2\}$ (where $2^2 = 1$ – only the remainder $\mod 3$ matters when you do arithmetic operations).

   (b) Using the formula from Problem 1, find how many solutions does the equation $x^2 + y^2 = 1$ have when $x, y \in \mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$. What about the solutions with x, y in $\mathbb{F}_5$?

   (c)* Generalize your answer for $\mathbb{F}_p$, where p is any prime number, and $\mathbb{F}_p = \{0, 1, 2, \ldots, p - 1\}$ (and the arithmetic operations are done $\mod p$). **Hint:** The answer depends on whether p gives remainder 1 or 3 when divided by 4, because depending on that, -1 is a square or not in $\mathbb{F}_p$.

3. Prove that the configuration of 7 points with the lines defined as in the picture is a projective plane, and it is the minimal projective plane (that is, you could not use fewer than 7 points to make one). (In this picture, the lines are: $\{a, b, e\}, \{c, d, e\}, \{c, g, b\}, \{a, g, d\}, \{a, c, f\}, \{b, d, f\}$).

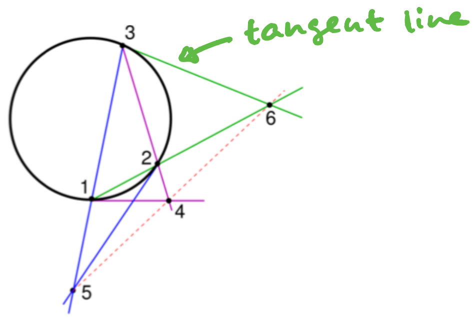


4. Derive Pappus' Theorem from Desargues' Theorem (see the notes for the statements of both theorems).

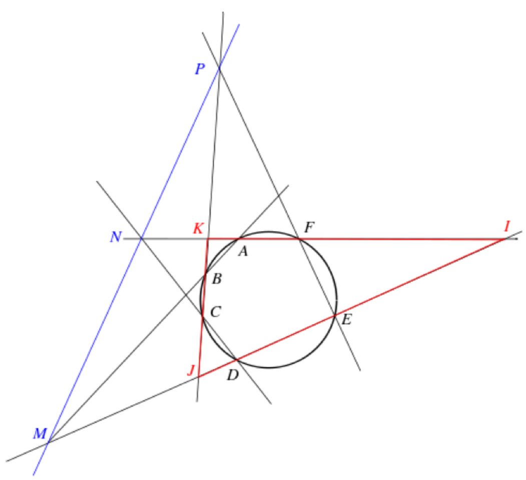
5. (a) State Pascal's theorem for a parabola and a hyperbola (see the lecture notes for the statement for the ellipse).

(b) Find other interesting Pasal's lines, given 6 points on an ellipse. (Hint: there are 60 possible lines, depending on how you name the points).

(c)* Prove the following degenerate case of Pascal's Theorem on a circle, and explain how this is indeed a case of Pascal's Theorem:



(d)* Prove Pascal's theorem in the case of 6 points on a *circle*.



(This is also a case of Pascal's Theorem: intersection points of the opposite sides of a hexagon inscribed in a circle lie on the same line)