UBC Math Circle 2021 Problem Set 4

1. Consider tic-tac-toe on a torus (aka the surface of a donut). This can be imagined by having the sides wrap around like in space invaders. For example:

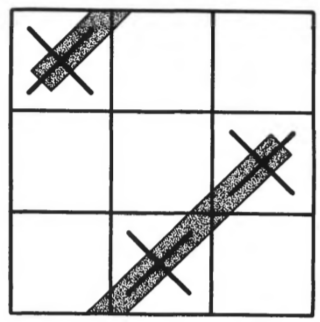


Figure 2.2 These Xs are three-in-a-row if the board is imagined to represent a torus.

From The Shape of Space by Jeffrey Weeks

   (a) Martin is bored and fills out all the squares of a 3×3 grid with X's and O's. Adam, who is passing by, has unusually good ears, and notes that Martin lifted his pencil exactly n times. Find, with proof, the number of (toroidal) tic-tac-toes on the board.

   (b) Adam decides to join in on the fun, and agrees to play Martin in a game of tic-tac-toe on a torus. Adam and Martin alternate writing down X's and O's, with Adam going first. Which player, if any, has a winning strategy?

2. Prove for all n that

$$n! = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n.$$

3. (a) We say a real number is 7-free if it does not include a 7 in its decimal expansion. (Numbers with finite decimal expansions like 6 may have a second decimal expansion, e.g. $6 = 5.999\ldots$. In that case, we take the finite decimal expansion to be its canonical decimal expansion.)

   Show that there exists $k \in \mathbb{N}$ such that for all $x > 0$, at least one of $x, 2x, \ldots, kx$ is not 7-free.

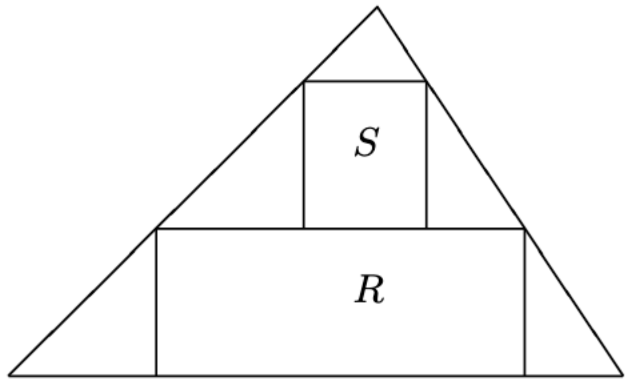
   (b) Consider the series $\sum_{n\in\mathbb{N}} \frac{1}{n}$ and $\sum_{\substack{n\in\mathbb{N} \\ n \text{ 7-free}}} \frac{1}{n}$.

   Which (if any) of the series converge? Which (if any) of the series diverge?

   (c) Give a second proof of the statement in part (a).

4. A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

5. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:



Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all acute triangles and R, S over all rectangles as above.