UBC Math Circle 2021 Problem Set 4

1. Consider tic-tac-toe on a torus (aka the surface of a donut). This can be imagined by having the sides wrap around like in space invaders. For example:

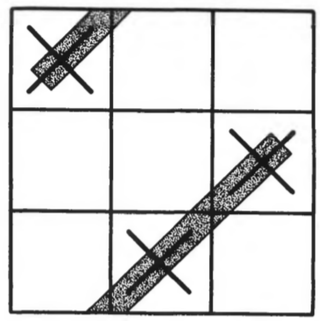


Figure 2.2 These Xs are three-in-a-row if the board is imagined to represent a torus.

From The Shape of Space by Jeffrey Weeks

(a) Martin is bored and fills out all the squares of a 3×3 grid with X's and O's. Adam, who is passing by, has unusually good ears, and notes that Martin lifted his pencil exactly n times. Find, with proof, the number of (toroidal) tic-tac-toes on the board.

Solution: Let a denote the number of X's and b denote the number of O's. Note that an X requires two pencil strokes while an O requires one. So we have $a + b = 9$ and $2a + b = n$. We obtain $a = n - 9$ and $b = 18 - n$.

Check that every square is in 4 "lines" or possible tic-tac-toes. Since there are 9 squares and every line contains 3 squares, it follows that there are $\frac{4 \cdot 9}{3} = 12$ lines or possible tic-tac-toes.

Every pair of distinct squares determines a unique line, and every line that is not a tic-tac-toe has either 2 X's and 1 O or 2 O's and 1 X. Therefore, the quantity ab counts the number of non-tic-tac-toe lines twice.

So the number of tic-tac-toes on the board must be

$$12 - \frac{ab}{2} = 12 - \frac{(n-9)(18-n)}{2}.$$

(b) Adam decides to join in on the fun, and agrees to play Martin in a game of tic-tac-toe on a torus. Adam and Martin alternate writing down X's and O's, with Adam going first. Which player, if any, has a winning strategy?

Solution: By a standard strategy-stealing argument, Adam has a strategy that never loses. (If Martin had a strategy that always won, then Adam could emulate that strategy after an arbitrary first move by pretending that the move Martin played after that was the real first move, etc. and therefore always win, a contradiction.)

By part (a), the number of tic-tac-toes on a completed board with 5 X's and 4 O's is always $12 - \frac{20}{2} = 2$. So there are no ties. So Adam's never-losing strategy is also an always-winning strategy.

2. Prove for all n that

$$n! = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n.$$

Solution: Note that the left-hand-side of the equation is $n!$, which is the number of permutations of n elements. We may associate the permutation $a_1, \ldots, a_n$ with the surjective function $f : i \mapsto a_i$, so there is a bijection between permutations of n elements and surjective functions $f : [n] \to [n]$.

Now, we just need to show that the right-hand-side counts the number of surjective functions from $[n]$ to $[n]$. First, we replace k with $n - k$ and use the fact that $\binom{n}{k} = \binom{n}{n-k}$:

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n &= \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (n-k)^n \\
&= n^n - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)^n \\
&= n^n - \left(\binom{n}{1} (n-1)^n - \binom{n}{2} (n-2)^n + \cdots + (-1)^n \binom{n}{n-1} 1^n \right).
\end{aligned}$$

The total number of functions from $[n]$ to $[n]$ is n^n (this corresponds with the term in the sum where $k = n$). Let A_i be the set of functions $f : [n] \to [n] \setminus \{i\}$. The number of functions that are not surjections is $|A_1 \cup \cdots \cup A_n|$. Note that

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = \binom{n}{k} (n-k)^n.$$

First we choose the k indices $i_1, \ldots, i_k$ that are not in the image of $f : [n] \to [n]$ (this gives $\binom{n}{k}$), then we choose $f(x)$ from the set $[n] \setminus \{i_1, \ldots, i_k\}$ for each $x \in [n]$ (this gives $(n-k)^n$). Then, inclusion-exclusion gives the desired sum for $|A_1 \cup \cdots \cup A_n|$.

Since the left and right sides of the equation count the same thing, they must be equal.

Polynomial solution: We first prove the following identity for polynomials over $\mathbb{R}$

$$n!\prod_{j=1}^{n} x_j = \sum_{k=1}^{n} \sum_{1\le i_1<\ldots<i_k\le n} (-1)^{n-k}(x_{i_1}+x_{i_2}+\cdots+x_{i_k})^n \tag{1}$$

For example, for $k = 3$ we have the following (it's best to try this example while following the proof below)

$$6xyz = (x+y+z)^3 - (x+y)^3 - (y+z)^3 - (x+z)^3 + x^3 + y^3 + z^3$$

At $x_1 = x_2 = \cdots = x_n = 1$ you will get the combinatorial identity in the question.

Proof of (1): Let $P(x_1,\ldots,x_n)$ be the righthandside polynomial of (1). Since $P(x_1,\ldots,x_n) = 0$ when any $x_i = 0$, we must have $x_i \mid P$ for every $1 \le i \le n$. Hence because $x_1,\ldots,x_n$ are irreducible in $\mathbb{R}[x_1,\ldots,x_n]$, we must have $\prod_{i=1}^n x_i \mid P$. But note that $\prod_{i=1}^n x_i$ has total degree n; and because of (1), $P = 0$ or it has total degree at most n. Hence, $P = 0$ or $P(x_1,\ldots,x_n) = c\prod_{i=1}^n x_i$ where $c \ne 0$. The latter must be true because $\prod_{i=1}^n x_i$ can only come from the expansion of $(x_1+\cdots+x_n)^n$ from the right hand side and it must be that the coefficient $c = n!$

What is the relation between this solution and the previous one?

3. (a) We say a real number is 7-free if it does not include a 7 in its decimal expansion. (Numbers with finite decimal expansions like 6 may have a second decimal expansion, e.g. $6 = 5.999\ldots$. In that case, we take the finite decimal expansion to be its canonical decimal expansion.)

   Show that there exists $k \in \mathbb{N}$ such that for all $x > 0$, at least one of $x, 2x, \ldots, kx$ is not 7-free.

   Solution: We may assume without loss of generality that $1 \le x < 10$. Note then that $70 \le \lceil \frac{70}{x} \rceil x < (\frac{70}{x}+1)x < 80$. So $\lceil \frac{70}{x} \rceil x$ is not 7-free.

   Since for $1 \le x < 10$ the quantity $\lceil \frac{70}{x} \rceil$ is a positive integer ≤ 70, taking $k = 70$ will work.

   You might also wonder what the smallest k satisfying is. I believe it's 42, but I have never formally proved it.

   (b) Consider the series $\sum_{n\in\mathbb{N}} \frac{1}{n}$ and $\sum_{\substack{n\in\mathbb{N} \\ n \text{ 7-free}}} \frac{1}{n}$.

   Which (if any) of the series converge? Which (if any) of the series diverge?

   Solution: $\sum_{n\in\mathbb{N}} \frac{1}{n} \ge 1 + (\frac{1}{2}+\frac{1}{2}) + (\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}) + \cdots \ge 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$

$\sum_{\substack{n\in\mathbb{N} \\ n \text{ 7-free}}} \frac{1}{n}$ surprisingly converges by a comparison test against a geometric series, see `https://en.wikipedia.org/wiki/Kempner_series`.

(c) Give a second proof of the statement in part (a).

Solution: Let x be any positive real number. Remember we may assume without loss of generality that $1 \le x < 10$.

If k is such that all of $x, 2x, \ldots, kx$ are 7-free, then $\sum_{i=1}^{k} \frac{1}{10k} \le \sum_{i=1}^{k} \frac{1}{\lfloor kx \rfloor} \le \sum_{\substack{n\in\mathbb{N} \\ n \text{ 7-free}}} \frac{1}{n} < \infty$.

Choosing k large enough such that $\sum_{i=1}^{k} \frac{1}{10k} > \sum_{\substack{n\in\mathbb{N} \\ n \text{ 7-free}}} \frac{1}{n}$ will work (and such k exists because the harmonic series diverges).

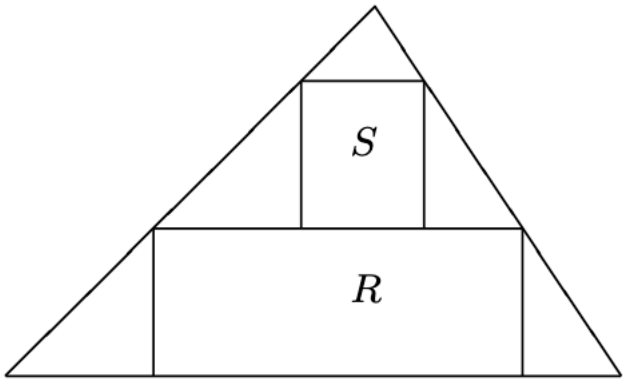
4. A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \ldots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

Solution: Since the sum of scores $2N(7.4) = \frac{74N}{5}$ is an integer, it follows that N is a multiple of 5. Therefore, the sum of scores is a multiple of 74, and so is even.

Sort the scores from lowest to highest, then assign every $(2n-1)$th student to one group, and every $(2n)$th student to the other. Since every score occurred at least once, the difference in the score sums of these two groups will only arise via pairs of the $(2n-1)$th and $(2n)$th students having a pair of scores in the form $(a, a+1)$. (The remaining pairs of $(2n-1)$th and $(2n)$th students will have pairs of scores in the form (a, a).)

Since the sum of scores is even, the number of these bad pairs must also be even. Swapping the students in half of these bad pairs will give a division of the class into two groups with equal numbers of students such that the score sums of both groups are equal. Therefore, both groups have exactly N students and the average score for each group is exactly 7.4.

5. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown:

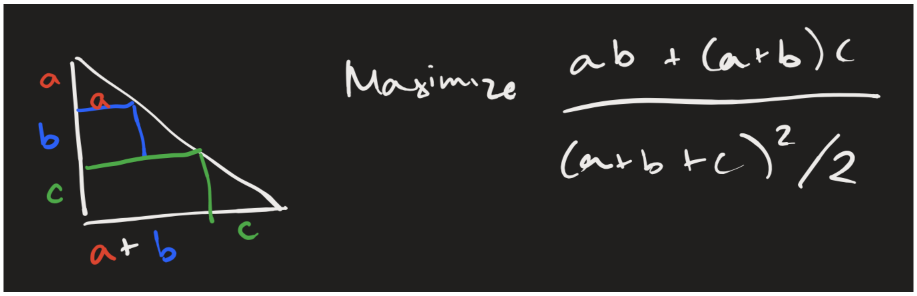


Let $A(X)$ denote the area of polygon X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all acute triangles and R, S over all rectangles as above.

Solution: It's first equivalent to maximize the quantity over isosceles triangles, by sliding the top vertex along a line parallel to the bottom edge.

Then, it's equivalent to maximize the quantity over an isosceles right triangle, by stretching the vertical axis as needed.

It is then equivalent to maximizing the quantity when looking at half the triangle, after drawing a line from the vertex to the middle of the hypotenuse cutting our isosceles right triangle in half.



Maximizing $\frac{2ab+2ac+2bc}{(a+b+c)^2}$ (as in the diagram) over nonnegative a, b, c is the same as minimizing the quantity $\frac{a^2+b^2+c^2-ab-ac-bc}{(a+b+c)^2}$. (Why?)

But $a^2+b^2+c^2-ab-ac-bc = \frac{1}{2}((a-b)^2+(b-c)^2+(a-c)^2)$, and so is minimized when $a = b = c$.

So the maximum ratio is $\frac{6a^2}{9a^2} = \frac{2}{3}$.