UBC Math Circle 2021 Problem Set 5

1. Let $P(x)$ be a real polynomial such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that there exist real polynomials $f(x)$ and $g(x)$ such that $P(x) = f(x)^2 + g(x)^2$.

> **Solution:** See https://www2.cms.math.ca/Competitions/COMC/examarchive/comc2014-officia
> pdf, C4 part (c).

2. Prove or disprove: for every $k \geq 1$, if $\mathbb{N}$ is coloured with k colours, then there must exist a monochromatic triple $(x, y, z) \in \mathbb{N}^3$ satisfying

$$x + y = 3z.$$

> **Solution:** We can colour $\mathbb{N}$ by elements of $(\mathbb{Z}/5\mathbb{Z})^\times$ as follows: for any $n \in \mathbb{N}$, write $n = 5^k m$ where m is coprime to 5, and assign n the colour the residue class of m modulo 5.
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> Suppose, for a contradiction, that there existed a monochromatic solution to $x + y = 3z$. Then write $x = 5^i a$ where i is maximal, and $y = 5^j b$, $z = 5^k c$ similarly.
>
> So $5^i a + 5^j b = 3 \cdot 5^k c$, where a, b, c are all congruent modulo 5 to the same nonzero residue. Let $l = \min\{i, j, k\}$.
>
> Then $5^{i-l} a + 5^{j-l} b = 3 \cdot 5^{k-l} c$. This implies that $5^{i-l} + 5^{j-l} \equiv 3 \cdot 5^{k-l} \pmod 5$ (after inverting by $a \equiv b \equiv c \pmod 5$). Note at least one of $i - l, j - l, k - l$ is zero. We can simply check over all $2^3 - 1 = 7$ possible cases (there are 7 cases since each 5^d for $d = i - l, j - l, k - l$ is congruent to either 0 or 1 modulo 5, depending on whether or not $d = 0$, and at least one of $i - l, j - l, k - l$ is 0) that there is no solution to this congruence equation.
>
> So we've constructed a 4-colouring of $\mathbb{N}$ with no monochromatic solution to $x + y = 3z$.

3. Let five points on a circle be labelled A, B, C, D, E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$. Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

> **Solution:** See https://www2.cms.math.ca/Competitions/CMO/archive/sol2018.pdf, 2.

4. Do there exist two weighted dice (with faces numbered from 1 to 6) such that the sum of the dice in a random roll is uniformly distributed in $\{2, 3, \ldots, 12\}$?

Solution: Let X, Y be the random variables associated with the outcomes of the rolls. Given a random variable X attaining only finitely many positive integer values, define the polynomial $\phi_X = \sum_{k=1}^{\infty} P(X = k)x^{k-1}$.

Note then that $\phi_X \phi_Y = \phi_{X+Y}$, after expanding out the product.

Since X attains values only in the set $\{1, \ldots, 6\}$, and since $X = 6$ must be attainable (in order to have $X+Y = 12$ be possible), we see that ϕ_X is a polynomial of degree 5. So it must have some real root. Similarly, ϕ_Y must have a real root. Then we cannot have $\phi_X \phi_Y = \phi_{X+Y} = \frac{1}{11}\sum_{k=1}^{11} x^k = \frac{1}{11}x\frac{x^{11}-1}{x-1}$, because all roots of the right hand side are the non-real 11th roots of unity and the root 0 with multiplicity 1, contradicting that the left hand side has at least two real roots (counting with multiplicity).

5. A *partition* of n is a weakly-sorted list of positive integers $(\lambda_1, \ldots, \lambda_\ell)$ whose sum is n. Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

Solution: See `https://math.stackexchange.com/questions/54961/the-number-of-partition`

There is both a solution constructing an explicit bijection, and a solution using generating functions.

6. Let $p > 3$ be a prime. Show that $x^2+x+1 \equiv 0 \pmod{p}$ has a solution iff $p \equiv 1 \pmod{3}$.

Solution: Suppose that $x^2 + x + 1 \equiv 0 \pmod{p}$ has a solution x_0. Then $x_0^3 \equiv 1 \pmod{p}$ and $x_0 \not\equiv 1 \pmod{p}$. Hence $\mathrm{ord}_p(x_0) = 3$ (or, if you don't like orders, 3 is the least positive integer n such that $x_0^n \equiv 1 \pmod{p}$). Because we know $x_0^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem, we must have $3 \mid p - 1$, i.e. $p \equiv 1 \pmod{3}$. Conversely, let $p \equiv 1 \pmod{3}$ be a prime, and let g be a primitive root modulo p. Then $x_0 := g^{(p-1)/3}$ satisfies $x_0^3 \equiv 1 \pmod{p}$ and trivially $x_0 \not\equiv 1 \pmod{p}$. Hence x_0 must be the root of $x^2 + x + 1 \equiv 0 \pmod{p}$.