UBC Math Circle 2022 Problem Set 3

1. Let $g : \mathbb{C} \to \mathbb{C}, \omega \in \mathbb{C}, a \in \mathbb{C}, \omega^3 = 1$, and $\omega \neq 1$. Show that there is one and only one function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(\omega z + a) = g(z), z \in \mathbb{C},$$

and find the function f.

2. Let f satisfy the functional equation

$$f(x)^2 = 1 + xf(x+1)$$

and the inequalities

$$\frac{x+1}{2} \leq f(x) \leq 2(x+1)$$

for all $x \geq 1$. Prove that $f(x) = x + 1$.

3. Let $f(x) = x^2 + 2022x + 1$. Define $f^{\circ n} = f \circ f \circ \ldots \circ f$ (n times). Prove that $f^{\circ n}$ has at least two real roots.

4. For every $n \geq 0$, find all polynomials $f(x) \in \mathbb{Z}[x]$ such that for all $x \in \mathbb{C} \setminus \{0\}$,

$$f\left(x + \frac{1}{x}\right) = x^n + \frac{1}{x^n}.$$

Your solution may be written in the form of a recurrence.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y$$

for all $x, y \in \mathbb{R}$.