UBC Math Circle 2022 Problem Set 4

1. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum \alpha_i = n$. Prove that

   (a) The number of compositions of n is 2^{n-1}.

   (b) The total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

   (c) For $n \geq 2$, the number of compositions of n with an even number of even parts is equal to 2^{n-2}.

2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

3. N cells are chosen on a rectangular grid. Let a_i is number of chosen cells in i-th row, b_j is number of chosen cells in j-th column. Prove that

$$\prod_i a_i! \cdot \prod_j b_j! \leq N!$$

4. You are given an unbiased fair coin C. Can you use C to simulate a biased coin C' which produces heads with probability p such that on average C is flipped twice? (i.e. either come up with a procedure which simulates C' and flips C twice on average or prove that no such procedure exists).

5. The Fibonacci numbers may be defined by the recurrence

$$F_0 = 0,\ F_1 = 1,$$

   and

$$F_n = F_{n-1} + F_{n-2}$$

   for $n > 1$.

   Show that

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

   for all $n \geq 1$ and $m \geq 0$. (Possible solution hint: observe that for $n > 1$, the number of possible ways of tiling a $1 \times (n-1)$ rectangle with monominos and dominos is equal to F_n.)