UBC Math Circle 2022 Problem Set 5

1. Find all solutions to the equation $E + V + F = G + 2$ where E, V, F, G are positive integers and E, V, F all divide G.

2. A population on a graph is an assignment of positive integers to each vertex. A *perfect population* has the property that the population of each vertex is exactly $1/2$ of the sum of the neighbouring populations. Find all perfectly populated (finite) graphs.

3. Let n be a positive odd integer. There are n computers and exactly one cable joining each pair of computers. You are to colour the computers and cables such that no two computers have the same colour, no two cables joined to a common computer have the same colour, and no computer is assigned the same colour as any cable joined to it. Prove that this can be done using n colours. What about when n is even?

4. Two pyramids with common base $A_1A_2A_3A_4A_5A_6A_7$ and vertices B and C are given. The edges $BA_i, CA_i (i = 1, \ldots, 7)$, the diagonals of the common base and the segment BC are coloured in either red or blue. Prove that there exists a triangle whose sides are colored in one and the same color.

5. For $k \in \mathbb{Z}_{\geq 0}$, a *proper k-colouring* of a graph G with vertex set V and edge set E is a map $\kappa : V \to \{1, \ldots, k\}$ so that for all edges $uv \in E$, $\kappa(u) \neq \kappa(v)$. Let $\chi_G(k)$ denote the number of proper k-colourings of G.

   (a) Show that $\chi_G(k)$ is a polynomial in k. χ_G is known as *the chromatic polynomial.*

   (b) What are the chromatic polynomials of the path and complete graphs on n vertices? (The path P_n has n vertices labelled $1, \ldots, n$ with an edge between each pair of vertices labelled i and $i + 1$. The complete graph K_n has n vertices and an edge between every pair of distinct vertices.)

   (c) An *acyclic orientation* of G is an assignment of a direction to each edge of G so that there are no directed cycles (i.e. there is no way to go from a vertex to itself by following directed edges). Show that the number of acyclic orientations of G is given by $|\chi_G(-1)|$.