UBC Math Circle 2022 Problem Set 6

1. (a) Prove that for any nonempty subsets A and B of $\mathbb{R}$ we have

$$|A+B| \geq |A| + |B| - 1,$$

where $A + B := \{a + b \mid a \in A, b \in B\}$.

(b) Prove that for any prime p and nonempty subsets A and B of $\mathbb{Z}/p\mathbb{Z}$ we have

$$|A+B| \geq \min\{p,\ |A| + |B| - 1\},$$

where $A + B := \{a + b \pmod p \mid a \in A, b \in B\}$.

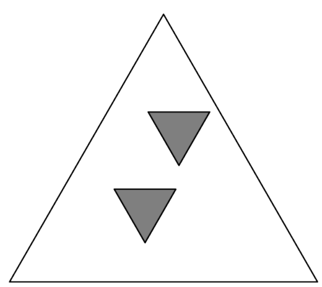
(Hint: You may use the fact that Combinatorial Nullstellensatz holds over $\mathbb{Z}/p\mathbb{Z}$.)

2. Prove that for any partition of the positive integers into a finite number of parts, one of the parts contains three integers x, y, z with $x + y = z$.

(Hint: It may be helpful to know Ramsey's theorem, which may be stated as follows. For any positive integer k, there is a positive integer n such that any gathering of n people contains either k mutual friends or k mutual strangers.)

3. For a prime p and a given integer n let $\nu_p(n)$ denote the exponent of p in the prime factorisation of $n!$. Given $d \in \mathbb{N}$ and $\{p_1, p_2, \ldots, p_k\}$ a set of k primes, show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \leq i \leq k$.

4. An equilateral triangle Δ of side length $L > 0$ is given. Suppose that n equilateral triangles with side length 1 and with non-overlapping interiors are drawn inside Δ, such that each unit equilateral triangle has sides parallel to Δ, but with opposite orientation. (An example with $n = 2$ is drawn below.)



Prove that

$$n \leq \frac{2}{3}L^2.$$

5. We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.

(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.

(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.