UBC Math Circle 2022 Problem Set 7

1. Given two triangles ΔABC and $\Delta A'B'C'$ with the same centroid, prove that one can construct a triangle with sides equal to the segments AA', BB', and CC'.

2. Let M be the midpoint of the side BC in ΔABC. Let E and F be the tangent points of the incircle and the sides CA and AB, respectively. Let the angle bisectors of $\angle B$ and $\angle C$ intersect the line EF at X and Y, respectively. Prove that ΔMXY is equilateral if and only if $\angle A = 60°$.

3. Inside of convex quadrilateral $ABCD$ is a point M such that $\angle AMB = \angle ADM + \angle BCM$ and $\angle AMD = \angle ABM + \angle DCM$. Prove that

$$AM \cdot CM + BM \cdot DM \geq \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$

4. The incircle of triangle $\triangle ABC$ with center I is tangent to the sides AB and BC at points C_1 and A_1, respectively. Let M be the midpoint of AC, and N be the midpoint of the arc ABC in the circumcircle of $\triangle ABC$. Let P be the projection of M over C_1A_1; show that I, P, N are collinear.

5. The incircle of triangle $\triangle ABC$ with center I has points of tangency D, E, and F. Let M be the foot of the perpendicular from D to EF, and let P be on DM such that $DP = MP$. If H is the orthocenter of BIC, prove that PH bisects EF.