UBC Math Circle 2022 Problem Set 9

1. Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences a_1, a_2, a_3, $\ldots$ of nonnegative real numbers satisfying

$$\sum_{n=1}^{\infty} a_k = 1.$$

2. Prove that the orthocenter, the centroid, and the circumcenter of any triangle are collinear. This line is the Euler Line of the triangle. Prove also that the distance from the centroid to the orthocenter is twice its distance from the circumcenter.

3. Consider the function $f \colon \mathbb{N} \to \mathbb{N}$ that satisfies the following conditions:

   1. For any natural number m, $f(m) \leq 3m$.
   2. $v_2(m+n) = v_2(f(m) = f(n))$ for any two natural numbers m, n.[1]

   Show that for any natural number $a \in \mathbb{N}$, there exists a unique number $b \in \mathbb{N}$ such that $f(b) = 3a$.

4. Let $V = \mathbb{Z}^n$ denote the n-dimensional integer lattice and let $\{w_1, w_2, \ldots, w_n\} \subset V$ be a set of n linearly independent integer vectors. Define $W \subset V$ to be the set of all integer linear combinations of the elements of $\{w_1, w_2, \ldots, w_n\}$. Construct a set V/W with elements of the form $v + W$ for $v \in V$ such that $u + W = v + W$ iff $u - v \in W$. Prove that $|V/W|$ is precisely the volume of the parallelotope P spanned by the vectors $\{w_1, w_2, \ldots, w_n\}$. (Hint: First prove that the volume of P is the sum $\sum_{i=0}^{n} \frac{|m_i|}{2^i}$ where m_i is the set of lattice points in an $n-i$-face of P but not in any $n-i-1$-face of P. For example, if $n = 3$ then m_0 is the set of lattice points inside the parallelopiped but not on any face of P, m_1 is the set of lattice points on a face of P but not on an edge of P, m_2 is the set of lattice points on an edge of P but not on any vertex of P, and m_3 is the set of lattice points on a vertex of P. Next show that $\sum_{i=0}^{n} \frac{|m_i|}{2^i} = |V/W|$.)

5. Given any positive real number ε, prove that, for all but finitely many positive integers v, any graph on v vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)

[1] $v_2(n)$ is the exponent of 2 in the factorization of n.